JOURNAL OF COMBINATORIAL THEORY 3, 25-33 (1967)

On the Inequality $\sum x_i y_i \geqslant 1/n \sum x_i \cdot \sum y_i$ and the van der Waerden Permanent Conjecture*

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Communicated by Gian-Carlo Rota

ABSTRACT

This paper consists of two parts. In Part I necessary and sufficient conditions are given for the inequality of the title to hold (Theorem I). In Part II an application is made to the van der Waerden permanent conjecture which yields a larger class of matrices than known heretofore for which the conjecture holds (Theorem II).

I. THE INEQUALITY

1. *Introduction:* In the book on inequalities by Hardy, Littlewood, and Pólya [1, p. 43], it is shown that if $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are similarly ordered sequences of real numbers, then (Tchebycheff)

$$\sum_{i=1}^{n} x_i y_i \geqslant \frac{1}{n}\Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{i=1}^{n} y_i\Big). \tag{1}$$

The inequality (1) can be valid for sequences that are not similarly ordered and our main result (Theorem I) gives necessary and sufficient conditions on the sequences $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ in order that (1) should hold. Let x, y be n-dimensional column vectors and let J_n be the n-square matrix with all entries equal to $1/n$. Let $K = I - J_n$ where I is the n-square identity matrix. Then (1) is equivalent to

$$(Kx, y) \geqslant 0 \tag{2}$$

where $(\cdot\,,\,\cdot)$ is the inner product. Let e be the n-dimensional column vector with all entries equal to 1. An n-square real matrix A is said to be *positive semidefinite* if the quadratic form (x, Ax) is non-negative for all real x or, equivalently, if the symmetric matrix $A + A'$ (where prime denotes transpose) is positive semidefinite.

* This work was supported by the United States Atomic Energy Commission.

2. *Main Theorem:*

THEOREM I. *In order that $(Kx, y) \geqslant 0$, it is necessary and sufficient that either $y = Ax + ae$ or $x = Ay + ae$ where a is a real number and A is a real positive semidefinite matrix whose row and column sums are zero. The inequality is strict unless $(A + A')\,x = 0$ (or $(A + A')\,y = 0$).*

PROOF OF THEOREM I: We first observe that $K' = K$, $Ke = 0$, and $KA = AK = A$ if A has row and column sums equal to 0. Therefore, if $y = Ax + ae$ where A has the properties in the theorem, we have

$$(Kx, y) = (x, K(Ax + ae)) = (x, Ax) \geqslant 0.$$

The inequality is strict unless $(x, Ax) = 0$ or, equivalently, $(x, (A + A')\,x) = 0$ since A is real. Since $A + A'$ is positive semidefinite and symmetric, $(x, (A + A')\,x) = 0$ implies $(A + A')\,x = 0$. This proves the sufficiency part of the theorem. To prove the necessity will require the following theorem due to Hardy, Littlewood, and Pólya [1, pp. 46–49]. We recall that an n-square matrix D is *doubly stochastic* if it has non-negative entries and has row and column sums equal to 1.

THEOREM (H-L-P). *Let x, y be n-dimensional column vectors such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. A necessary and sufficient condition for the existence of a doubly stochastic matrix D such that $y = Dx$ is that the system of inequalities*

$$\sum_{i=1}^{k} y_i \leqslant \sum_{i=1}^{k} x_i\,, \qquad k = 1, 2, \ldots, n$$

be satisfied, with equality holding for $k = n$.

COROLLARY. *Let x, y satisfy the hypothesis of the preceding theorem and, in addition, suppose $x_1 > x_n$. Then there exist real numbers λ and μ and a matrix D such that $\lambda > 0$, D is doubly stochastic and $\lambda y + \mu e = Dx$.*

PROOF OF COROLLARY: We need only show there exist real numbers $\lambda > 0$ and μ satisfying

$$\lambda \sum_{i=1}^{k} y_i + k\mu \leqslant \sum_{i=1}^{k} x_i\,, \qquad k = 1, 2, \ldots, n$$

with equality for $k = n$. Solving for μ in the equation when $k = n$ and substituting into the remaining inequalities we obtain

$$\frac{\lambda}{n} \sum_{i=k+1}^{n} \Big(\sum_{j=1}^{k} y_j - k y_i \Big) \leqslant \frac{1}{n} \sum_{i=k+1}^{n} \Big(\sum_{j=1}^{k} x_j - k x_i \Big), \qquad k = 1, 2, \ldots, n - 1.$$

The right-hand side is positive for each k since $x_1 > x_n$ and hence this system is satisfied for λ positive and sufficiently small.

We will also need the following lemma.

LEMMA 1. *Let T be a linear transformation on Euclidean n-space such that $(x, Tx) \geqslant 0$ for all x in a linear subspace N. Then there exists a linear transformation S such that $Sx = Tx$ for all $x \in N$ and $(x, Sx) \geqslant 0$ for all x. Moreover, if $x \in N$ and $(T + T')\, x \in N$, then $S'x = T'x$.*

PROOF: Let P be the self-adjoint orthogonal projection onto N and define $S = TP - PT' + PT'P$. Since $Px = x$ for $x \in N$, it is clear that $Sx = Tx$ for $x \in N$. For arbitrary x we have

$$\begin{aligned}(x, Sx) &= (x, TPx) - (x, PT'x) + (x, PT'Px)\\ &= (x, TPx) - (TPx, x) + (TPx, Px)\\ &= (TPx, Px)\\ &\geqslant 0.\end{aligned}$$

If $x \in N$ and $(T + T')\, x \in N$, then

$$\begin{aligned}S'x &= PT'x - TPx + PTPx\\ &= PT'x - Tx + PTx\\ &= P(T + T')\, x - Tx\\ &= (T + T')x - Tx\\ &= T'x.\end{aligned}$$

Now let x, y satisfy $(Kx, y) \geqslant 0$. If $x_1 = x_2 = \cdots = x_n = a$, then $x = Ay + ae$ with $A = 0$. Hence assume not all the x_i are equal. Let P and Q be permutation matrices such that the entries in Px and Qy are in non-ascending order. Then by the corollary, there exist real numbers $\lambda > 0$ and μ and a doubly stochastic matrix D such that

$$\lambda Qy + \mu e = DPx$$

and therefore

$$\lambda y + \mu e = Q'DPx.$$

Since $(Kx, y) \geqslant 0$, $(Kx, e) = (x, Ke) = 0$, and $\lambda > 0$, it follows that

$(Kx, \lambda y + \mu e) \geqslant 0$. Let α, β be real numbers. Utilizing the fact that $Q'DP$ is doubly stochastic we compute

$$\begin{aligned}(\alpha x + \beta e, Q'DP(\alpha x + \beta e)) &= \alpha^2(x, Q'DPx) + 2\alpha\beta(x, e) + n\beta^2 \\ &\geqslant \alpha^2(J_n x, Q'DPx) + 2\alpha\beta(x, e) + n\beta^2 \\ &= (\alpha^2/n)(x, e)^2 + 2\alpha\beta(x, e) + n\beta^2 \\ &= (1/n)(\alpha(x, e) + n\beta)^2 \\ &\geqslant 0.\end{aligned}$$

Thus (u, $Q'DPu) \geqslant 0$ for u in the linear space N spanned by x and e. By Lemma 1, there exists a matrix B such that $Bx = Q'DPx$, $Be = e$, $B'e = e$, and B is positive semidefinite. Let

$$A = (1/\lambda)(B - J_n)$$

and

$$a = (1/\lambda)\left((1/n)\sum_{i=1}^{n} x_i - \mu\right).$$

Then

$$y = Ax + ae,$$

where A has row and column sums equal to 0. All that is left is to show that A is positive semidefinite, which is equivalent to showing that the symmetric matrix $B + B' - 2J_n$ is positive semidefinite. The condition $Be = B'e = e$ implies that $B + B'$ and $2J_n$ commute and therefore are simultaneously diagonalizable. The eigenvalues of $2J_n$ are 2 and 0, the eigenvalue 2 being simple with a one-dimensional eigenspace spanned by e, which is also an eigenvector for $B + B'$ corresponding to the eigenvalue 2. It follows from these remarks that $B + B' - 2J_n$ is positive semidefinite if and only if $B + B'$ is. Therefore A is positive semidefinite which completes the proof.

If $y = Ax + ae$ where $Ae = A'e = 0$ and A is positive semidefinite, then $y = Bx + be$ where $B = A + cJ_n$, $b = a - (c/n)\sum_{i=1}^{n} x_i$, and $Be = B'e = ce$. If $c \geqslant 0$, an argument analagous to that given in the proof of Theorem I shows that B is positive semidefinite if and only if A is. Thus the essential requirement in Theorem I is that A is positive semidefinite and has all row and column sums equal to the same non-negative constant, the value zero being chosen simply as a convenient normalization. We state this equivalent formulation of Theorem I as the following corollary.

COROLLARY. *In order that* $(Kx, y) \geqslant 0$, *it is necessary and sufficient* *that either $y = Ax + ae$ or $x = Ay + ae$ where a is a real number and A is a real positive semidefinite matrix whose row and column sums are all equal and non-negative. The inequality is strict unless $(A + A')\,x$ (or $(A + A')\,y$) is a scalar multiple of e.*

II. An Application to the van der Waerden Permanent Conjecture

1. *Statement of Theorem II:* As an application of Theorem I, we obtain a new result concerning the van der Waerden conjecture on the permanent of a doubly stochastic matrix. The permanent of an n-square complex matrix $A = (a_{ij})$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma i}$$

where the summation extends over all permutations σ in the symmetric group of degree n. More generally, for an $m \times n$ matrix A and $k \leqslant m, n$, let $P_k(A)$ be the sum of the permanents of all k-square submatrices of A.

Generalized van der Waerden Conjecture: If A is an n-square doubly stochastic matrix and $A \neq J_n$, then $P_k(A) > P_k(J_n)$, $1 < k \leqslant n$.

The van der Waerden conjecture is the case $k = n$ which, to our knowledge at this writing, is still unresolved. In section 3 we will prove the following.

Theorem II. *If A is an n-square doubly stochastic, normal matrix not equal to J_n and if all the eigenvalues of A lie in the wedge in the complex plane defined by $|\arg z| \leqslant \pi/2k$, then $P_k(A) > P_k(J_n)$, $1 < k \leqslant n$.*

Theorem II generalizes the result demonstrated previously [2] that if A is an n-square doubly stochastic, positive semidefinite, symmetric matrix and $A \neq J_n$, then $P_n(A) > P_n(J_n)$.

2. *Notation:* For positive integers k and n, $G_{k,n}$ will denote the set of functions with domain the integers $\{1, 2,..., k\}$ and range $\{1, 2,..., n\}$. Members of $G_{k,n}$ will be denoted by $f, g, h,...$. $F_{k,n}$ will denote the subset of $G_{k,n}$ consisting of those functions which are one-to-one. (Thus $F_{k,n}$ is empty for $k > n$.) Members of $F_{k,n}$ will denoted by $\sigma, \tau,...$. If A is an $m \times n$ matrix and $f \in G_{p,m}$, $g \in G_{q,n}$, the symbols (f, A), (A, g), (f, A, g) will denote, respectively, the $p \times n$, $m \times q$, $p \times q$ dimensional matrices whose entries in the i-th row and j-th column are $a_{f(i),j}$, $a_{i,g(j)}$, and $a_{f(i),g(j)}$.

3. *Cauchy-Binet Identity and Proof of Theorem II:* The principal tool

we will use in the proof of Theorem II is an easily proved form of the Cauchy-Binet identity:

THEOREM. Let A and B be $m \times n$ and $n \times m$ matrices, respectively. Then

$$P_m(AB) = \frac{1}{m!} \sum_{f \in G_{m,n}} P_m(A, f)\, P_m(f, B). \tag{3}$$

More generally, if A and B are $m \times n$ and $n \times p$ matrices, respectively, and $k \leqslant m, p$ then

$$P_k(AB) = \frac{1}{k!} \sum_{f \in G_{k,n}} P_k(A, f)\, P_k(f, B). \tag{4}$$

PROOF:

$$P_m(AB) = \sum_{\sigma \in F_{m,m}} \prod_{i=1}^{m} \sum_{k=1}^{n} a_{ik} b_{k\sigma(i)}$$

$$= \frac{1}{m!} \sum_{\sigma, \tau \in F_{m,m}} \prod_{i=1}^{m} \sum_{k=1}^{n} a_{\tau(i)k} b_{k\sigma(i)}$$

$$= \frac{1}{m!} \sum_{\sigma, \tau \in F_{m,m}} \sum_{f \in G_{m,n}} \prod_{i=1}^{m} a_{\tau(i)f(i)} b_{f(i)\sigma(i)}$$

$$= \frac{1}{m!} \sum_{f \in G_{m,n}} \sum_{\sigma, \tau \in F_{m,m}} \prod_{i=1}^{m} a_{\tau(i)f(i)} \prod_{i=1}^{m} b_{f(i)\sigma(i)}$$

$$= \frac{1}{m!} \sum_{f \in G_{m,n}} P_m(A, f)\, P_m(f, B).$$

Similarly,

$$P_k(AB) = \frac{1}{(k!)^2} \sum_{\sigma \in F_{k,m}} \sum_{\tau \in F_{k,p}} P_k(\sigma, AB, \tau)$$

which by the previous result, since $(\sigma, AB, \tau) = (\sigma, A)(B, \tau)$,

$$= \frac{1}{(k!)^3} \sum_{\sigma \in F_{k,m}} \sum_{\tau \in F_{k,p}} \sum_{f \in G_{k,n}} P_k(\sigma, A, f)\, P_k(f, B, \tau)$$

$$= \frac{1}{(k!)^3} \sum_{f \in G_{k,n}} \sum_{\sigma \in F_{k,m}} P_k(\sigma, A, f) \sum_{\tau \in F_{k,p}} P_k(f, B, \tau)$$

$$= \frac{1}{k!} \sum_{f \in G_{k,n}} P_k(A, f)\, P_k(f, B).$$

If in (4) we specialize $B = (b_{ij})$ to $b_{ij} \equiv 1$, we obtain

$$\frac{1}{k!} \sum_{f \in G_{k,n}} P_k(A, f) = E_k(r_1, \ldots, r_m) \tag{5}$$

and dually

$$\frac{1}{k!} \sum_{f \in G_{k,m}} P_k(f, A) = E_k(c_1, \ldots, c_n) \tag{6}$$

where r_i is the i-th row sum of A, c_j is the j-th column sum of A and E_k is the k-th elementary symmetric function of its indicated arguments.

Again, if we take $m = n = p$ and $B =$ identity matrix,

$$P_k(A) = \frac{1}{k!} \sum_{f \in G_{k,n}} P_k(A, f)\, P_k(f, I).$$

Thus if the n^k-dimensional vectors $\{P_k(A, f)\}$ and $\{P_k(f, I)\}$ are related by an affine transformation of the type described in Theorem I or its corollary, it will follow from that theorem and (5) and (6) above that

$$P_k(A) \geqslant \frac{k!}{n^k} \binom{n}{k} E_k(r_1, \ldots, r_n). \tag{7}$$

If A is doubly stochastic, then (7) becomes

$$P_k(A) \geqslant P_k(J_n) = \frac{k!}{n^k} \binom{n}{k}^2.$$

Suppose then that A is doubly stochastic and apply the generalized Cauchy-Binet formula (4) to $\mathrm{I} \cdot (A, f)$:

$$P_k(A, f) = \frac{1}{k!} \sum_{g \in G_{k,n}} P_k(I, g)\, P_k(g, A, f); \tag{8}$$

from (5) and (6) it follows that the n^k-square matrix

$$W_k(A) = \frac{1}{k!} \{P_k(g, A, f)\}$$

is doubly stochastic, and so by the corollary to Theorem I it will follow that

$$P_k(A) \geqslant P_k(J_n) \quad \text{if} \quad W_k(A) + (W_k(A))' \text{ is positive semidefinite}$$

The following properties of $W_k(A)$ are easily verified:

$$(W_k(A))' = W_k(A'), \tag{9}$$

$$\overline{W_k(A)} = W_k(\bar{A}), \tag{10}$$

$$W_k(AB) = W_k(A)\, W_k(B). \tag{11}$$

Now let A be an n-square doubly stochastic, normal matrix whose eigenvalues $d_1, d_2, \ldots, d_n$ satisfy $|\arg d_i| \leqslant \pi/2k$, $i = 1, 2, \ldots, n$. Let U be a unitary matrix such that $A = UDU^*$ where $D = \operatorname{diag}\{d_1, d_2, \ldots, d_n\}$ and $U^* = \bar{U}'$. Then by (9), (10), (11),

$$W_k(A) = W_k(UD)(W_k(U))^*.$$

Moreover, since the permanent is a homogeneous function of each row (or each column) of a matrix, it follows that

$$P_k(f, UD, g) = P_k(f, U, g) \prod_{i=1}^{k} d_{g(i)},$$

and therefore

$$W_k(A) = W_k(U)\, M W_k(U)^*,$$

where M is the n^k-square diagonal matrix

$$\operatorname{diag}\left\{\ldots, \prod_{i=1}^{k} d_{g(i)}, \ldots\right\}.$$

Finally, we have

$$W_k(A) + (W_k(A))' = W_k(U)(M + \bar{M})(W_k(U))^*,$$

and since the entries of M are products of k of the d_i, the condition $|\arg d_i| \leqslant \pi/2k$ implies that $M + \bar{M}$ has non-negative entries and consequently $W_k(A) + (W_k(A))'$ is positive semidefinite. Thus $P_k(A) \geqslant P_k(J_n)$. To show that strict inequality holds (for $k > 1$) unless $A = J_n$, we again appeal to the corollary of Theorem I, which in this instance states that equality holds only if $P_k(A, f) + P_k(A', f) = \alpha$ is a constant independent of $f \in G_{k,n}$. Let B be the $n \times k$ matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then by (5)

$$\frac{1}{k!}\sum_{f\in G_{k,n}} (P_k(A,f)+P_k(A',f))\,P_k(f,B)=\frac{\alpha}{k!}\sum_{f\in G_{k,n}} P_k(f,B)=0; \tag{12}$$

on the other hand, by (4), the left-hand side of (12) is equal to $P_k(AB)+P_k(A'B)$. The matrix AB has as its first two columns the difference of the first two columns of A and all the other entries are equal to 1 since A is doubly stochastic. Similarly, the first two columns of $A'B$ are the difference of the first two rows of A and all other entries are 1. $P_k(AB)$ may be evaluated by a Laplace expansion on the first two columns to obtain

$$P_k(AB)=(k-2)!\binom{n-2}{k-2}\sum_{i\neq j}(a_{i1}-a_{i2})(a_{j1}-a_{j2}), \tag{13}$$

and in a similar manner

$$P_k(A'B)=(k-2)!\binom{n-2}{k-2}\sum_{i\neq j}(a_{1i}-a_{2i})(a_{1j}-a_{2j}). \tag{14}$$

If in (13) and (14) we perform the summation on one of the indices and use the doubly stochastic constraint, we obtain

$$P_k(AB)+P_k(A'B)=-(k-2)!\binom{n-2}{k-2}\sum_{i=1}^{n}(a_{1i}-a_{2i})^2+(a_{i1}-a_{i2})^2. \tag{15}$$

The vanishing of the right-hand side of (15) implies the first two rows of A are identical and the first two columns of A are identical. In a similar manner we can show that any pair of rows of A are identical and the symmetrical pair of columns of A are identical. Clearly, this can occur only if all entries of A are equal, i.e., $A=J_n$.

References

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2. M. Marcus, and M. Newman, Inequalities for the Permanent Function, *Ann. of Math.* **75** (1962), 47–62.